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Lie algebra application to mobile robot control: a tutorial

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SUMMARY
Lie algebra is an area of mathematics that is largely used by electrical engineer students, mainly at post-graduation level in the control area. The purpose of this paper is to illustrate the use of Lie algebra to control nonlinear systems, essentially in the framework of mobile robot control. The study of path following control of a mobile robot using an input-output feedback linearization controller is performed. The effectiveness of the nonlinear controller is illustrated with simulation examples.

KEYWORDS: Lie algebra; Input-output linearization; Nonlinear systems; Mobile robots.

1. INTRODUCTION

It is well known that linear algebra concepts, as well as matrix theory, are essential in the approaches to handle the linearization control problem of nonlinear systems, specifically when their relative degree is not well defined. However, the relationship between the control of nonlinear systems and the matrix theory is not always easy to understand. Teaching Lie algebra to electrical engineering students takes place mainly at post-graduation level, and the skills needed for its application are obtained usually in electronic databases, books and by the analysis of scientific papers that sometimes are neither sufficiently clear nor easy to understand. The purpose of this paper is to illustrate the use of Lie algebra concepts in the linearization and control of a nonlinear system (i.e. mobile robot).

Last decade was a period of immense activity in deriving control algorithms, namely in the control of the mobile robots motion. One of the difficulties concerns planning robot trajectories.[1–3] Another difficulty concerns path following control. There are several recent works dedicated to this subject,[4–8] where Lie algebra shows to be essential. To a better understanding of the Lie algebra application it is possible to seek information in the classic references of Hall[9] and Varadarajan[10] and in the textbooks on nonlinear systems of Isidori,[11] Nijmeijer[12] and Khalill.[13]

In this paper we study the path following control of a mobile robot, making use of an input-output feedback linearization controller.

The paper is organized in the following manner. Key topics concerning a nonholonomic robot dynamics and its state-space representation are presented in Section 2. Also in this section, the controlled linearization feedback input-output and the determination/existence of relative degrees are described, that the concept of Lie brackets and Lie derivatives are also presented. Section 3 presents the kinematic model and the constraint equations of the mobile robot in study, as well as the dynamic model equations. In Section 4, the output equations that are essential to the control algorithm are determined. This section also presents simulation results that illustrate the effectiveness of the input-output feedback linearization controller in controlling mobile robot trajectories.

2. DYNAMICS EQUATIONS AND THEORETIC FORMULATION

Consider a nonholonomic mobile robot with n generalized coordinates q subject to m constraints (assuming that $m<n$) whose dynamics equations of motion are described by:

$$M(q)\ddot{q}+V(q,\dot{q})=B(q)\tau-A^T(q)\lambda \qquad (1)$$

where $V(q,\dot{q})=C(q,\dot{q})\dot{q}$, $M(q)\in R^{n\times n}$ is the inertia matrix, $C(q,\dot{q})\in R^{n\times n}$ is the centriptal and coriolis forces matrix, $B(q)\in R^{n\times(n-m)}$ is the input transformation matrix, $A^T(q)$ is a Jacobian matrix, $\tau\in R^{(n-m)}$ is the input vector, and $\lambda\in R^m$ is the vector of constraint forces. The m constraint equations of the mechanical system can be written in the form

$$w(q,\dot{q})=0.$$

If a constraint equation is in the form $W_i(q)=0$, is named holonomic, otherwise it is a kinematics constraint named nonholonomic.

2.1. *Classification of nonholonomic systems*

It is important to know the type of system's motion constraints. Some concepts and mathematical formulations that allow to reach this purpose will be presented. Suppose that there are k holonomic and $m-k$ nonholonomic constraints, all of which can be written in the form of[14]

$$A(q)\dot{q}=0 \qquad (2)$$

where $A(q)\in R^{m\times n}$ is a full rank matrix. Let $s_1,\ldots,s_{n-m}$ be a set of *smooth** and linearly independent vector fields in the null space of $A(q)$, $\mathcal{N}(A)$, i.e.

$$A(q)s_i(q)=0 \quad i=1,\ldots,n-m.$$

Let $S(q)$ be the full rank matrix made up of these vectors

$$S(q)=[s_1(q)\ \ldots\ s_{n-m}(q)]$$

* Continuously differentiable.

and Δ the distribution spanned by these vector fields

$$\Delta(q)=span\{s_1(q),\ldots,s_{n-m}(q)\} \tag{3}$$

thus, it follows that $dim\Delta(q)=rank\ S(q)$ and any $\dot{q}$ satisfying equation (2) belongs to Δ.

Definition 1: *For two vector fields f and g, the Lie bracket is a third vector field defined by:*

$$[f,g](q)=\frac{\partial g}{\partial q}f(q)-\frac{\partial f}{\partial q}g(q).$$

It is obvious that $[f,g]=-[g,f]$ and $[f,g]=0$ *for constant vector fields f and g. Also the Jacobi identity,*

$$[h,[f,g]]+[f,[g,h]]+[g,[h,f]]=0.$$

The following notation is commonly used in Lie bracket representation:

$$ad_f^0 g(q)=g(q)$$
$$ad_f g(q)=[f,g](q)$$
$$ad_f^k g(q)=[f,ad_f^{k-1}g](q),\quad k>1$$

Definition 2: *A distribution Δ is involutive if it is closed under Lie bracket operation, that is, if*

$$g_1\in\Delta \quad\text{and}\quad g_2\in\Delta \quad\Rightarrow\quad [g_1,g_2]\in\Delta.$$

Then, we analyse whether Δ distribution is or not involutive. Let Δ^* be the smallest involutive distribution containing Δ, in this case $dim(\Delta)\leq dim(\Delta^*)$. According to Campion et al.[15] there are three possible cases: (1) for $k=m$, i.e. all the constraints are holonomic, Δ is involutive; (2) for $k=0$, i.e. all the constraints are nonholonomic, Δ^* spans the entire space; (3) for $0<k<m$, the k constraints are integrable and k components of the generalized coordinates may be eliminated from the motion equations. In the last case $dim(\Delta^*)=n-k$.

However, we may be more precise, and distinguish among holonomic, and nonholonomic constraints. To verify the type of constraints it is necessary computing repeated Lie brackets of the vector fields $s_1,\ldots,s_{n-m}$ of Δ (or of the system $\dot{q}(t)=\sum_{i=1}^{n-m}s_i v_i(t)=S(q)v(t)$).

As noted by Luca,[16] "The level of bracketing needed to span R^n is related to the complexity of the motion planning problem. For this reason, we give below a classification of nonholonomic systems based on the sequence and order of Lie brackets in the corresponding accessibility algebra."

Definition 3: *The filtration generated by the distribution Δ (3) is defined*[16] *as the sequence $\{\Delta_i\}$ with*

$$\Delta_i=\Delta_{i-1}+[\Delta_1,\Delta_{i-1}],\quad i\geq 2$$

where

$$\Delta_1=\Delta$$

and

$$[\Delta_1,\Delta_{i-1}]=span\{[s_j,\gamma]\,|\,s_j\in\Delta_1,\ \gamma\in\Delta_{i-1}\},\ j=1,\ldots,n-m$$

Note that $\Delta_i\subseteq\Delta_{i+1}$. Also, from the Jacobi identity follows that $[\Delta_i,\Delta_j]\subseteq[\Delta_1,\Delta_{i+j-1}]\subseteq\Delta_{i+j}$.

A filtration is regular[16] *in a given neighbourhood V of q_0 if $dim\Delta_i(q)=dim\Delta_i(q_0)$, $\forall q\in V$.*

For a regular filtration, if $dim\Delta_{i+1}=dim\Delta_i$, then Δ_i is involutive and $\Delta_{i+j}=\Delta_i$ for all $j\geq 0$. Since $dim\Delta_1=n-m$ and $dim\Delta_i\leq n$, the termination condition takes place after m steps, i.e. it agrees with the number of original kinematics constraints.

If the filtration generated by a distribution Δ is regular, it is possible to define the *degree of nonholonomy* of Δ as the smallest integer κ that verifies the condition $dim\Delta_{\kappa-1}=dim\Delta_\kappa$. Note that the verification of this condition implies that $\kappa\leq m+1$.

The previous conditions for holonomy, partial nonholonomy and complete nonholonomy may be rewritten as follows: (1) for $\kappa=1$, i.e. $dim\Delta_\kappa=n-m$, all the constraints are holonomic; (2) for $2\leq\kappa\leq m$ and if $dim\Delta_\kappa=n$, all the constraints are nonholonomic; (3) for $2\leq\kappa\leq m$ and if $(n-m)+1\leq dim\Delta_\kappa<n$, the constraints are partially nonholonomic.

2.2. *State space representation*

Now consider the mechanical system given by (1) and (2) and let k of the m constraints be holonomic. Since the constrained velocity is always in the null space of $A(q)$, it is possible to define[14] $n-m$ velocities $v(t)=[v_1 v_2\ldots v_{n-m}]^T$ such that for all t

$$\dot{q}=S(q)v(t). \tag{4}$$

The previous equation represents the kinematics of a mechanical system (in this case a mobile robot), where $S(q)$ is basically a Jacobian matrix that converts velocities from a mobile coordinates system to velocities in a cartesian coordinates system.

Differentiating equation (4) with respect to t, after substituting the result ($\ddot{q}$) in equation (1), and finally multiplying the result by S^T gives

$$S^T(MS\dot{v}(t)+M\dot{S}v(t)+V)=S^T B\tau \tag{5}$$

taken into consideration that $S^T A^T\lambda=0$, since the matrix S spans $\mathcal{N}(A)$. Considering the state vector

$$x=\begin{bmatrix} q\\ v\end{bmatrix} \tag{6}$$

and based on equations (4) and (5) is attained the state equation:

$$\dot{x}=\begin{bmatrix} Sv\\ f_2\end{bmatrix}+\begin{bmatrix} 0\\ (S^TMS)^{-1}S^TB\end{bmatrix}\tau \tag{7}$$

where $f_2=(S^TMS)^{-1}(-S^TM\dot{S}v-S^TV)$. Assuming that the number of system inputs is greater or equal to the difference between the number of generalized coordinates and the number of independent constraints of the mechanical system ($r\geq n-m$), and that $(S^TMS)^{-1}S^TB$ has rank $n-m$, the following nonlinear feedback[7] can be applied:

$$\tau=((S^TMS)^{-1}S^TB)^+(u-f_2), \tag{8}$$

where $(.)^+$ denotes a generalized inverse of (.). The state equation can be rewritten to the form:

$$\dot{x}=f(x)+g(x)u \tag{9}$$

where

$$f(x)=\begin{bmatrix} S(q)v \\ 0 \end{bmatrix} \text{ and } g(x)=\begin{bmatrix} 0 \\ I \end{bmatrix}$$

2.3. Input-output feedback linearization

Consider the following single-input and single-output system:

$$\begin{aligned}\dot{x}&=f(x)+g(x)u\\ y&=h(x)\end{aligned} \tag{10}$$

where $x\in R^n$ represents the state vector, $u\in R$ is the control input and $y\in R$ is the output. f and g are vector fields (henceforward called *functions*), h is a function, and all are nonlinear and assumed to be differentiable.

In the input-output feedback linearization problem the question is to find out, if it exists, a state feedback control law

$$u=\alpha(x)+\beta(x)v$$

and a transformation of state variables

$$z=T(x) \tag{11}$$

that transforms the nonlinear system into an equivalent linear one. The variable v is an external input, $\alpha(x)$ and $\beta(x)$ are scalar algebraic functions with $\beta(x)\neq 0$.

The transformation matrix $T(x)$ must be invertible, such that $x=T^{-1}(z)$ can be feasible, and since the derivatives of z and x should be continuous, both $T(.)$ and $T^{-1}(.)$ must be continuously differentiable. A continuously differentiable map with a continuously differentiable inverse is known as a **diffeomorphism**.

Nonholonomic systems have unique properties. Thus the system (9) is controllable if all of its constraints are nonholonomic and its equilibrium point $x=0$ can be made Lagrange stable but cannot be made asymptotically stable by a smooth state feedback[15] (see stability definitions in Appendix C).

It can be stated that a system with nonholonomic constraints is not input-state linearizable, and it may be input-output linearizable if a proper set of output equations are chosen.[7,14] It is noteworthy that in input-state linearization the state equation is completely linearized, while in the input-output linearization, where the input-output map is linearized, the state equation may be only partially linearized.

The knowledge of the relative degree, ρ, is one of the conditions for the application of feedback linearization methods. The system's relative degree is the smallest order of derivatives of output, y, that explicitly depends on the input, u.

Thus, by differentiation of (10), it is found that:

$$\dot{y}=\frac{\partial h}{\partial x}\dot{x}=\frac{\partial h}{\partial x}[f(x)+g(x)u]\overset{\text{def}}{=}L_f h(x)+L_g h(x)u$$

where

$$L_f h(x)=\frac{\partial h}{\partial x}f(x)$$

is defined as the Lie derivative of a scalar function $h(x)$ with respect to a vector function $f(x)$ or along $f(x)$. This is the common notion of derivative of h along the trajectories of the system $\dot{x}=f(x)$. To proceed with the calculation of the relative degree it is necessary to know the following set of Lie algebra expressions:

$$L_g L_f h(x)=\frac{\partial(L_f h)}{\partial x}g(x)$$

$$L_f^2 h(x)=L_f L_f h(x)=\frac{\partial(L_f h)}{\partial x}f(x)$$

$$L_f^k h(x)=L_f L_f^{k-1}h(x)=\frac{\partial(L_f^{k-1}h)}{\partial x}f(x)$$

$$L_f^0 h(x)=h(x).$$

If $L_g h(x)\neq 0$, then $\rho=1$. If $L_g h(x)=0$, then $\dot{y}=L_f h(x)$ is independent of u. Calculating the second derivative of y, denoted by $y^{(2)}$, holds:

$$y^{(2)}=\frac{\partial(L_f h)}{\partial x}[f(x)+g(x)u]=L_f^2 h(x)+L_g L_f h(x)u$$

Once again, if $L_g L_f h(x)\neq 0$, then $\rho=2$. If $L_g L_f h(x)=0$ then $y^{(2)}=L_f^2 h(x)$ that is independent of u. Proceeding the calculation, the definition of *relative degree* ρ is obtained, since if $h(x)$ fulfil

$$\begin{aligned}&L_g L_f^{i-1}h(x)=0, \quad i=1,2,\ldots,r-1\\ &L_g L_f^{r-1}h(x)\neq 0\end{aligned}$$

then u does not appear in the equations of $y, \dot{y}, \ldots, y^{(r-1)}$, and appears with a nonzero coefficient in the equation of $y^{(r)}$, i.e. $\rho=r$ and

$$y^{(\rho)}=L_r^{\rho}h(x)+L_g L_f^{\rho-1}h(x)u. \tag{12}$$

The foregoing equation shows that the system is input-output linearizable and described by the equation

$$y^{(\rho)}=v \tag{13}$$

if the following control law is chosen

$$u=\frac{1}{L_g L_f^{\rho-1}h(x)}[v-L_f^{\rho}h(x)].$$

From (13) it is concluded that the resulting linearized system, i.e. a system with input v and output y, is a chain of ρ integrators.

An important feature of the input-output feedback linearization method is the fact that it decomposes nonlinear system dynamics into an external part and an internal part. Since the external part consists of a linear relation between y and v (or equivalently, the controllability canonical form[17] between y and u) it is easy to design the input v so that the output y behaves as desired. Then, the question is whether the internal dynamics will also behave well or not, that is if the internal states will remain bounded. Since the control design must account for the whole dynamics (and therefore cannot tolerate the instability of internal dynamics), the internal behavior has to be addressed carefully. The problem of instability is usually overcome by means of using the zero dynamics of the system. Zero dynamics can be achieved by considering that the inputs and initial conditions of the system are chosen in such a way that the system output is identically *zero*.

The input-output linearization, is based on the application of transformation (11), that allow the transformed system to be presented as:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_{\rho-1} &= z_\rho \\
\dot{z}_\rho &= v \\
\dot{z}_{\rho+1} &= q_{\rho+1}(z) \\
&\vdots \\
\dot{z}_n &= q_n(z) \\
y &= z_1
\end{aligned} \tag{14}$$

where

$$z = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \\ \vdots \\ T_\rho(x) \\ T_{\rho+1}(x) \\ \vdots \\ T_n(x) \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{\rho-1} h(x) \\ T_{\rho+1}(x) \\ \vdots \\ T_n(x) \end{bmatrix} \tag{15}$$

From (14) it is verified that the latest $n-\rho$ equations are not observable by means of the output; they are, however, necessary to guarantee that they are stable. The functions $T_{\rho+1}(x), \ldots, T_n(x)$ can be chosen arbitrarily, as long as it is guaranteed that $T(x)$ is a diffeomorphism. Particularly, based on (14), it is possible to achieve $T_{\rho+1}(x), \ldots, T_n(x)$ functions, such that the latest $n-\rho$ equations are independent from the input u. From system (14) it can be verified that the first ρ equations exhibit an input-output behavior of an *integrator* of ρ order (such as presented earlier in the study of relative degrees).

Two situations must be distinguished: (1) when $\rho = n$, we have an input-state linearization and the state and output equations represent a linear system; (2) when $\rho < n$, we have an input-output linearization, and some state variables will exhibit a nonlinear dynamics.

To illustrate the application of the input-output linearization method, consider that the output equations are only functions of the position state variables, q, (see (6)). Since the number of the degrees of freedom of the system is instantaneously $n-m$, we may have in this case at most $n-m$ independent output equations:

$$y = h(q) = [h_1(q) \ldots h_{n-m}(q)]$$

Thus, to determine the relative degree it is necessary to differentiate the output, y, until the input u appears explicitly. Supposing that u appears, for the first time, in the second derivative of y, then the relative degree of the system is $\rho = 2$. Therefore, the transformation (11) allows to represent the system (15) in the form

$$z = T(x) = T(q) = \begin{bmatrix} h(q) \\ L_f h(q) \\ T_3(q) \\ \vdots \\ T_n(q) \end{bmatrix} = \begin{bmatrix} h(q) \\ \dfrac{\partial h}{\partial q}\dot{q} \\ T_3(q) \\ \vdots \\ T_n(q) \end{bmatrix} = \begin{bmatrix} h(q) \\ \Phi(q)v \\ T_3(q) \\ \vdots \\ T_n(q) \end{bmatrix}, \tag{16}$$

where $\dot{q} = S(q)v(t)$, $\Phi(q) = J_h(q)S(q)$ is the decoupling matrix, and $J_h = \dfrac{\partial h}{\partial q} \in R^{(n-m)\times n}$ is the Jacobian matrix.

Note that the functions $T_3(q), \ldots, T_n(q)$ were not yet defined. At this point the question is to analyse the internal dynamics of the system.

As noticed previously, the internal dynamics is the unobservable dynamics of the system (14). And to distinguish controllable and observable dynamics from unobservable dynamics, is commonly used the following notation (see reference [15]).

$$z = T(x) = \begin{bmatrix} \xi \\ \eta \end{bmatrix},$$

where

$$\xi = \begin{bmatrix} z_1 \\ \vdots \\ z_\rho \end{bmatrix}, \quad \eta = \begin{bmatrix} z_{\rho+1} \\ \vdots \\ z_n \end{bmatrix}.$$

Consequently, the system (14) is decomposed into two parts, where the unobservable part is η. It is obtained the socalled *normal form* of the system:

$$\begin{gathered}
\begin{bmatrix} \dot{\xi}_1 \\ \vdots \\ \dot{\xi}_{\rho-1} \\ \dot{\xi}_\rho \end{bmatrix} = \begin{bmatrix} \xi_2 \\ \vdots \\ \xi_\rho \\ a(\xi,\eta) + b(\xi,\eta)u \end{bmatrix} \\
\begin{bmatrix} \dot{\eta}_1 \\ \vdots \\ \dot{\eta}_{n-\rho} \end{bmatrix} = \begin{bmatrix} q_1(\xi,\eta) \\ \vdots \\ q_{n-\rho}(\xi,\eta) \end{bmatrix} \\
y = \xi_1
\end{gathered} \tag{17}$$

where

$$\begin{aligned}
a(\xi,\eta) &= L_f^\rho h(x) \\
b(\xi,\eta) &= L_g L_f^{\rho-1} h(x) \\
q_i(\xi,\eta) &= L_f \eta_i(x) \quad 1 \le i \le n-\rho
\end{aligned}$$

At this moment, it is already known the state transformation, that partially linearizes the system, and the control law.

However, it is still necessary to determine the η-vector elements:

$$\eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_{n-\rho} \end{bmatrix} = \begin{bmatrix} T_{\rho+1}(x) \\ \vdots \\ T_n(x) \end{bmatrix}.$$

Knowing that

$$\dot{\eta}_i = \frac{\partial T_j}{\partial x}\dot{x} = \frac{\partial T_j}{\partial x}(f(x)+g(x)u) = L_f T_j(x) + L_g T_j(x)u,$$

$$j = \rho+1, \ldots, n.$$

and so as to prevent that $\dot{\eta}$ doesn't depend explicitly on u, the T_j functions are chosen in such a way that

$$L_g T_j = \frac{\partial T_j}{\partial x} g(x) = 0. \tag{18}$$

To prove that the nonlinear system represented by equations (9) and (10) can indeed be transformed into the normal form (17), it has to be shown, not only that such transformation of coordinates exist, but also that it is a "true" state transformation. In other words, it has to be shown that a diffeomorphism

$$T(x) = [\xi_1 \ldots \xi_\rho \quad \eta_1 \ldots \eta_{n-\rho}]^T$$

can be "built" so that (17) is verified. To know if $T(x)$ is a diffeomorphism, is enough to examine if its Jacobians are invertible. It is, then, necessary to show that there are $\eta - \rho$ functions of η_i which are essential to complete the coordinates transformation.

From a practical point of view, explicitly finding the η vectors field, which is necessary to complete the transformation in normal form, forces us to solve the set of partial differential equations in η_i, such as indicated in equation (18).

Considering the aforementioned assumptions (i.e. y is only function of the state variable q, and $\rho=2$) the equation (16) can now be completed as explained in the following.

Considering the nonlinear system (9)

$$x = \begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} S(q)v \\ 0 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{g(x)} u$$

the equation (16) can be written in the normal form (17). First, is obvious that

$$\xi_1 = h(q)$$
$$\xi_2 = L_f h(q) = \Phi(q)v$$

Second, considering only the state variable q, the nonlinear system (9) satisfies the equation (18), since:

$$L_g T_j = \frac{\partial T_j}{\partial q} g(q) = \frac{\partial T_j}{\partial q_1}.0 + \frac{\partial T_j}{\partial q_2}.0 + \ldots + \frac{\partial T_j}{\partial q_n}.0 = 0,$$

$$\forall \quad j = \rho+1, \ldots, n$$

As the result of this equation is always zero, an arbitrary expression can be chosen for T_j (or η_i) to confirm the diffeomorphism. Thus, let us choose the solution $\eta_i = \tilde{h}_i$ $\forall$ $i = \rho+1, \ldots, n$. Then, equation (16) become:

$$z = T(x) = \begin{bmatrix} h(q) \\ L_f h(q) \\ \tilde{h}_1(q) \\ \vdots \\ \tilde{h}_{n-\rho}(q) \end{bmatrix},$$

which for $m = n-\rho$ can be simplified to

$$z = T(x) = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} h(q) \\ L_f h(q) \\ \tilde{h}(q) \end{bmatrix} = \begin{bmatrix} h(q) \\ \Phi(q)v \\ \tilde{h}(q) \end{bmatrix} \tag{19}$$

where $\tilde{h}(q) \in R^m$ is a vectorial function in such a way that the matrix $[J_h^T \, J_{\tilde{h}}^T]$ is full rank.

Considering that $T(x)$ is a diffeomorphism, the system under the new state variable z is characterized by

$$\dot{z}_1 = \dot{\xi}_1 = \frac{\partial h}{\partial q}\dot{q} = z_2 \tag{20}$$

$$\dot{z}_2 = \dot{\xi}_2 = \dot{\Phi}(q)v + \Phi(q)u = \upsilon \tag{21}$$

$$\dot{z}_3 = \dot{\eta} + J_{\tilde{h}} S v = J_{\tilde{h}} S (J_h S)^{-1} z_2. \tag{22}$$

See appendix A for the demonstration of equations (20)–(22).

The necessary and sufficient condition for input-output linearization is that the decoupling matrix has full rank.[12] $\Phi(x)$ can be obtained at the time the derivatives of y are calculated, as described in Section 4.1. $\Phi(x)$ is nonsingular if the rows of J_h are independent of the rows of $A(q)$.

From (21), we determine the state feedback

$$u = \Phi^{-1}(q)(\upsilon - \dot{\Phi}(q)v) \tag{23}$$

that leads to the input-output linearization and to the input-output decoupling, taking into consideration only the observable part of the system:

$$\dot{z}_1 = \dot{\xi}_1 = z_2$$
$$\dot{z}_2 = \dot{\xi}_2 = \upsilon$$
$$y = h(q) = z_1.$$

The internal dynamics associated with the input-output linearization corresponds to the last $(n-\rho)$ equations $\dot{\eta} = q(\xi, \ \eta)$ of the normal form (17). Generally, this dynamics depends on the output states ξ. However, we can define an intrinsic property of the nonlinear system by considering the system's internal dynamics when the control input is such that the output y is maintained at zero. The study of this so-called *zero-dynamics* will allow us to reach some conclusions about the stability of the internal dynamics.

The constraint that the output y is identically zero implies that all of its time derivatives are zero. Thus, the corresponding internal dynamics of the system, or zero-dynamics, describes motion restricted to the $(n-\rho)$ –

dimensional smooth surface (*manifold*) M_0 defined by $\xi=0$. In order that the system operates in zero-dynamics, i.e. to have the state X on the surface M_0, $X(0)$ must be on the surface, and the input, u, must be such that y stays at zero

$$y^{(\rho)}(t)=0 \tag{24}$$

Substituting (24) in (12), it can be concluded that u should adopt the expression

$$u_0(x)=-\frac{L_f^{\rho}h(x)}{L_gL_f^{\rho-1}h(x)}.$$

Using this input, and assuming that the system's initial state is on the surface, i.e. that $\xi=0$, the system dynamics can be simply written in the normal form as

$$\begin{aligned}\dot{\xi}&=0\\ \dot{\eta}&=q(0,\,\eta).\end{aligned} \tag{25}$$

By definition (25) is the *zero-dynamics* of nonlinear system constituted by the equations (9) and (10).[17]

Finally, for the system composed by equations (20)–(22) and by applying the definition (25), (i.e. substituting $z_1=0$ and $z_2=0$), it is found that for the system's *zero dynamics*

$$\dot{z}_3=J_{\bar{h}}S(J_hS)^{-1}\underbrace{z_2}_{0}\quad\Rightarrow\quad\dot{z}_3=0$$

which is Lagrange stable but not asymptotically stable.

3. EXAMPLE – DIFFERENTIAL MOBILE ROBOT

3.1. Kinematic model and constraint equations

Consider the dynamics of a two-wheeled mobile robot, which can move forward, and spin about its geometric center, as shown in Figure 1.

This robot is actuated by two wheels, $2b$ is the length of the axis between the wheels of the mobile robot and r is the radius of the wheels. $\{W\}$ is the inertial coordinates system (or world coordinates system) and $\{R\}$ is the coordinates system fixed to the mobile robot. P_0 is the origin of $\{R\}$ and is placed in the middle of the driving wheel axis. θ_r and θ_l denote the angles of rotation of the two wheels, right and left respectively (with respect to arbitrary initial states). The pose of the robot is given by the (x, y) position of its center and the heading angle ϕ. P_c is the center of mass of the robot with coordinates (x_c, y_c), and is placed in the X-axis at a distance d of P_0 and finally a is the length of the robot in the direction perpendicular to the driving wheel axis. The balance of the robot is maintained by a small castor whose effect we shall otherwise ignore. Thus, $q=(x_c, y_c, \phi, \theta_r, \theta_l)$ denotes the configuration of the system, i.e. the five generalized coordinates.

In the kinematic model it is supposed that in each contact exist a pure *rolling motion*, i.e. each wheel can roll in the direction in which it points and spin about its vertical axis, but cannot slide. Assuming that the velocity of P_0 must be in the direction of the axis of symmetry (X-axis) and the wheels must not slip, are obtained, with respect to P_c, the following constraints set:[18]

$$\dot{y}_c\cos\phi-\dot{x}_c\sin\phi-\dot{\phi}d=0 \tag{26}$$

$$\dot{x}_c\cos\phi+\dot{y}_c\sin\phi+b\dot{\phi}-r\dot{\theta}_r=0 \tag{27}$$

$$\dot{x}_c\cos\phi+\dot{y}_c\sin\phi-b\dot{\phi}-r\dot{\theta}_l=0 \tag{28}$$

These constraints can be rewritten in the form

$$A(q)\dot{q}=0$$

with

$$A(q)=\begin{bmatrix}-\sin\phi & \cos\phi & -d & 0 & 0\\ -\cos\theta & -\sin\phi & -b & r & 0\\ -\cos\phi & -\sin\phi & b & 0 & r\end{bmatrix}. \tag{29}$$

Considering the mobile robot kinematics, it comes for $S(q)$:

$$S(q)=[s_1(q),\,s_2(q)]=$$

$$\begin{bmatrix}c(b\cos\phi-d\sin\phi) & c(b\cos\phi+d\sin\phi)\\ c(b\sin\phi+d\cos\phi) & c(b\sin\phi-d\cos\phi)\\ c & -c\\ 1 & 0\\ 0 & 1\end{bmatrix}$$

that satisfies the equation $A(q)S(q)=0$, and where the constant $c=\frac{r}{2b}$. The kinematic model, given by (4) is

$$\dot{q}=S(q)v(t),\text{ with } v=\begin{bmatrix}\dot{\theta}_r\\ \dot{\theta}_l\end{bmatrix}.$$

At this stage the nonholonomy test is fulfilled with the purpose of determining the type of robot's kinematics constraints.

Using the filtration concept, introduced in Section 2.1, it follows that

$$\Delta_1=\Delta(q)=span\{s_1(q),\,s_2(q)\}$$

where $dim\ \Delta_1=rank[s_1(q),\,s_2(q)]=2$.

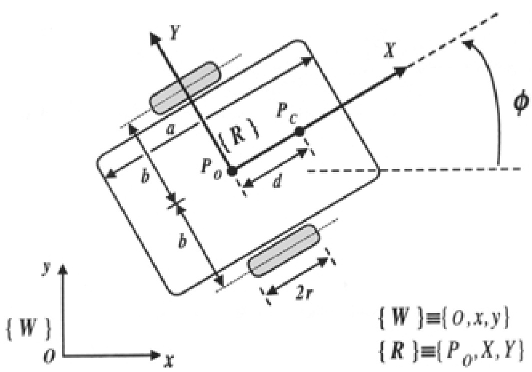

Fig. 1. Differential mobile robot.

Thus, computing the Lie bracket of $s_1(q)$ and $s_2(q)$, we have (see appendix-B for demonstration)

$$s_3(q)=[s_1(q), s_2(q)]=\frac{\partial s_2}{\partial q}s_1-\frac{\partial s_1}{\partial q}s_2=\ldots=\begin{bmatrix}-rc\ sin\ \phi\\ rc\ cos\ \phi\\ 0\\ 0\\ 0\end{bmatrix}.$$

Since $s_3(q)$ is not a linear combination of $s_1(q)$ and $s_2(q)$, then $s_3(q)$ is not in the distribution Δ_1 spanned by $s_1(q)$ and $s_2(q)$, and so at least one of the constraints is nonholonomic. Then the Δ_2 distribution assumes the form of $\Delta_2=span\{s_1(q), s_2(q), s_3(q)\}$ with $dim\ \Delta_2=rank\ [s_1(q), s_2(q), s_3(q)]=3$. Using the same calculation process presented for $s_3(q)$, the Lie brackets are calculated so as to obtain $s_4(q)$ and $s_5(q)$:

$$s_4(q)=[s_1(q), s_3(q)]=\begin{bmatrix}-rc^2\ cos\ \phi\\ -rc^2\ sin\ \phi\\ 0\\ 0\\ 0\end{bmatrix}, \text{ and}$$

$s_5(q)=[s_2(q), s_3(q)]=-s_4(q)$, as $s_5(q)$ is multiple of $s_4(q)$ then Δ_3 is uniquely $\Delta_3=span\{s_1(q), s_2(q), s_3(q), s_4(q)\}$, and $dim\ \Delta_3=rank[s_1(q), s_2(q), s_3(q), s_4(q)]=4$;

In the same way, the next Lie brackets combinations are:

$$s_6(q)=[s_1(q), s_4(q)]$$
$$=[rc^3\ sin\ \phi-rc^3\ cos\ \phi\ 0\ 0\ 0]^T=-c^2s_3(q)\in\Delta_3;$$
$$s_7(q)=[s_2(q), s_4(q)]=c^2s_3(q)\in\Delta_3;$$
$$s_8(q)=[s_3(q), s_4(q)]=0\in\Delta_3.$$

As $s_6(q)$ and $s_7(q)$ are multiples of $s_3(q)$, $s_8(q)$ is a linear combination of $s_3(q)$ and $s_4(q)$, and all of them are in the distribution Δ_3, then $\Delta_4=span\{s_1(q), s_2(q), s_3(q), s_4(q)\}=\Delta_3$ and $dim\Delta_4=4$.

As $dim\Delta_4=dim\Delta_3$, then Δ_3 is involutive and $\Delta_{3+j}=\Delta_3$ for all $j\geq 0$. From this result, we can conclude that the distribution spanned by $s_1(q)$, $s_2(q)$, $s_3(q)$ and $s_4(q)$ is involutive being all its vectors linearly independent and, hence,

$$\Delta^*=span\{s_1(q), s_2(q), s_3(q), s_4(q)\}.$$

The nonholonomy degree of Δ^* is $\kappa=3$, and the system is partially nonholonomic. From definition 3 (Section 2.1) it can be concluded that among the three constraints, two of them are nonholonomic.

Subtracting equation (28) from equation (27), the holonomic constraint is obtained,

$$\dot{\phi}=\frac{r}{2b}(\dot{\theta}_r-\dot{\theta}_l). \tag{30}$$

In conclusion, we have one holonomic constraint (30), and two nonholonomic constraints:

$$\dot{y}_c\ cos\ \phi-\dot{x}_c\ sin\ \phi-\dot{\phi}d=0$$

$$\dot{x}_c\ cos\ \phi+\dot{y}_c\ sin\ \phi-\frac{r}{2}(\dot{\theta}_r+\dot{\theta}_l)=0.$$

3.2. *Dynamic model*

To achieve the dynamics equations of the mobile robot it is necessary to calculate the Lagrangian so as to obtain the Lagrange equations of motion and, consequently, the M, V and B matrixes (referred to in Section 2) in a way to make easy the state-space representation.

The Lagrangian equations of motion[18] for the robot in study can be written as:

$$m\ddot{x}_c+md(\ddot{\phi}\ sin\ \phi+\dot{\phi}^2\ cos\ \phi)-\lambda_1\ sin\ \phi-cos\ \phi(\lambda_2+\lambda_3)=0$$

$$m\ddot{y}_c-md(\ddot{\phi}\ cos\ \phi-\dot{\phi}^2\ sin\ \phi)+\lambda_1\ cos\ \phi-\ sin\ \phi(\lambda_2+\lambda_3)=0$$

$$I\ddot{\phi}-md\ddot{y}_c\ cos\ \phi+md\ddot{x}_c\ sin\ \phi-d\lambda_1+b(\lambda_3-\lambda_2)=0$$

$$I_w\ddot{\theta}_r+\lambda_2r=\tau_r$$

$$I_w\ddot{\theta}_l+\lambda_3r=\tau_l$$

where

$$m=m_c+2m_w, \text{ and } I=I_c+2m_w(d^2+b^2)+2I_m+m_cd^2.$$

In the above equations, m_w is the mass of each driving wheel plus the rotor of its motor, m_c is the mass of the robot platform, I_c is the inertia of robot platform about a vertical axis through P_c (the center of mass), I_w, is the inertia of each wheel with the motor's rotor about the wheel axis, I_m, is the inertia about a defined axis in the plan of the wheel (perpendicular to the wheel axis), τ_r and τ_l are the torques of the right and left wheels, respectively.

The five motion equations can easily be written in the form of equation (1). The matrices $M(q)$, $V(q, \dot{q})$ and $B(q)$ are the following ones.[18]

$$M(q)=\begin{bmatrix}m & 0 & md\ sin\ \phi & 0 & 0\\ 0 & m & -md\ cos\ \phi & 0 & 0\\ md\ sin\ \phi & -md\ cos\ \phi & I & 0 & 0\\ 0 & 0 & 0 & I_w & 0\\ 0 & 0 & 0 & 0 & I_w\end{bmatrix}$$

$$V(q,\dot{q})=\begin{bmatrix}md\dot{\phi}^2\ cos\ \phi\\ md\dot{\phi}^2\ sin\ \phi\\ 0\\ 0\\ 0\end{bmatrix}\quad B(q)=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}\quad \tau=\begin{bmatrix}\tau_r\\ \tau_l\end{bmatrix}$$

and, finally, using the state variable $x=[x_c\ y_c\ \phi\ \theta_r\ \theta_l\ \dot{\theta}_r\ \dot{\theta}_l]^T$, the mobile robot dynamics can be represented in a state-space form of equation (7).

4. MOBILE ROBOT CONTROL

4.1. Output equation

The output variables, unlike the state equations that are uniquely determined by the dynamic characteristics of the system, are chosen in such a way that the tasks to be performed by the dynamic system can be conveniently specified, making easier the controller design.

Since the system has two inputs, any two-output variable may be chosen. However, if we intend to analyse the mobile robot path following, the two most important requirements are to follow the path with the smallest possible error and with a desired velocity. Thus, it is necessary to choose an output equation with two variables: (1) the shortest distance of the reference point on the mobile robot to the desired path; and (2) the forward velocity.

Consequently, to achieve "path following" it is necessary to choose appropriately h_1 and h_2. h_1 is defined as the shortest distance from the point P_c, on the mobile robot, to the desired path, and h_2 is defined as the forward velocity (of P_c along the X-axis), therefore the output vector is

$$y=h(x)=[h_1(q)\ h_2(v)]^T.$$

Let us consider a circular path as example. The expression of the distance from the point (x_c, y_c) to the path is easily obtained. Let P_g be the center of the circular path whose coordinates are denoted by (x_g, y_g) in the inertial referential, and let R be the radius of the circular path, hence h_1 and the forward velocity of the robot, h_2, can be chosen as follows:

$$h_1(q)=h_1(x_c, y_c, \phi)=\left|\sqrt{(x_c-x_g)^2+(y_c-y_g)^2}-R\right|$$

$$h_2(v)=\dot{x}_c \cos\phi+\dot{y}_c \sin\phi=\frac{r}{2}(v_1+v_2).$$

The following step consists of performing the system's input-output linearization. Assuming that $y(x)=[h_1(q)\ h_2(v)]^T$, it is necessary to obtain the relative degrees for $h_1(q)$ and $h_2(v)$ separately, so as to obtain for each of them the new external input variable v. Therefore, each of $y(x)$ elements is differentiated until each of its elements explicitly depend on the input u. Hence, the first derivative of y_1 holds:

$$\dot{y}_1=\frac{\partial h_1}{\partial x}\dot{x}=\frac{\partial h_1}{\partial x}[f(x)+g(x)u]=L_f h(x)+L_g h(x)u. \quad (31)$$

We need to use Lie algebra to calculate $L_f h(x)$ in (31). As h_1 depends only on q, and $L_g h(x)=0$ (as it can be verified by equation (9)), hence

$$\dot{y}_1=\frac{\partial h_1}{\partial x}\dot{x}=\frac{\partial h_1}{\partial q}\dot{q}=\Phi_1(q)v(t)$$

where $\Phi_1(q)=J_{h_1}S(q)$. As this derivative is not explicitly function of u it is necessary to continue the differentiation process, in function of q, resulting

$$\ddot{y}_1=L_f^2 h(x)+L_g L_f h(x)u \Rightarrow \ddot{y}_1=\frac{\partial(J_{h_1}Sv)}{\partial q}S(q)v+J_{h_1}(q)S(q)u=\upsilon_1.$$

As $\ddot{y}_1$ is already in function of u, it can be concluded that the relative degree of y_1 is two. In the same way, the first derivative of y_2, regarding that $h_2(v)$ is exclusively function of v, is expressed by

$$\dot{y}_2=\frac{\partial h_2}{\partial x}\dot{x}=\frac{\partial h_2}{\partial v}\dot{v}$$

Considering again equation (9) it can be observed that $\dot{q}=S(q)v$ and $\dot{v}=u$, and therefore

$$\dot{y}_2=\frac{\partial h_2}{\partial v}u=J_{h_2}u=\upsilon_2$$

being this last expression in function of the input u. It is then verified that the relative degree of y_2 is one. In conclusion, the y derivatives can be represented as:

$$\dot{y}_1=J^{h_1}(q)S(q)v(t)$$

$$\ddot{y}_1=\frac{\partial(J_{h_1}Sv)}{\partial q}S(q)v+J_{h_1}(q)S(q)u$$

$$\dot{y}_2=J_{h_2}u$$

At this stage it becomes necessary to determine the Jacobians as follows:

$$J_{h_1}(q)=\frac{\partial h_1}{\partial q}=\frac{1}{\sqrt{(x_c-x_g)^2+(y_c-y_g)^2}}\times$$

$$[(x_c-x_g)\ (y_c-y_g)\ 0\ 0\ 0]$$

$$J_{h_2}(v)=\frac{\partial h_2}{\partial v}=\begin{bmatrix}\frac{r}{2} & \frac{r}{2}\end{bmatrix}$$

The decoupling matrix of the system is given by $\Phi=\begin{bmatrix}J_{h_1}(q)S(q)\\ J_{h_2}\end{bmatrix}$ whose determinant has the following expression:

$$det(\Phi)=\frac{r^2 d}{2b}\frac{(\sin\phi(x_g-x_c)+\cos\phi(y_c-y_g))}{\sqrt{x_c^2-2x_cx_g+x_g^2+y_c^2-2y_cy_g+y_g^2}}.$$

The necessary and sufficient condition for the system to be input-output linearized and controllable is that $det(\Phi)\neq 0$. If this condition is verified, by applying the nonlinear feedback (equation (23)), we get a linearized and decoupled system in the following form:

$$\ddot{y}_1=\upsilon_1$$

$$\dot{y}_2=\upsilon_2$$

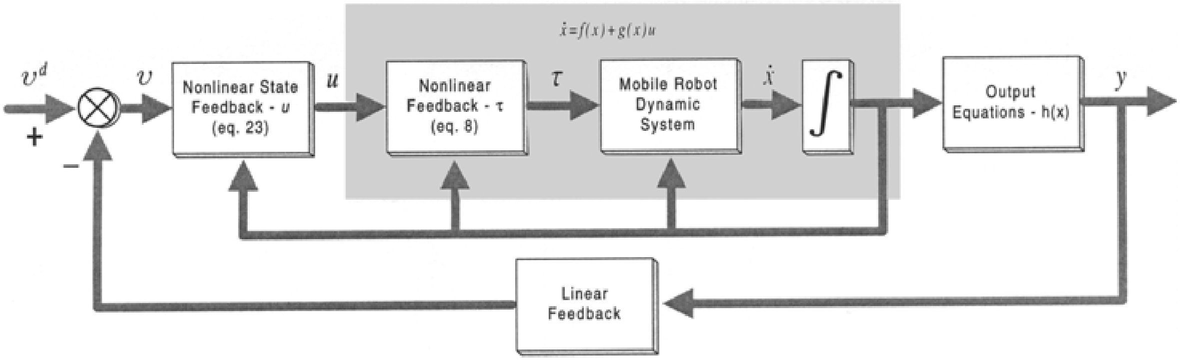

Fig. 2. Block diagram of the control system.

Note that we are going to design a linear feedback loop so that each subsystem becomes stable under an "ideal" conditions of operation.

4.2. Controller design

Finally, after obtaining all the conditions necessary to the implementation of the feedback control by input-output linearization, it is possible to design the controller. The path following control scheme[7] is presented in Figure 2.

In Figure 2, v^d represents the desired values for the outputs, h_1, and h_2. The nonlinear feedback (equation 8) allows us to cancel the nonlinearity in the robot dynamics so that the state equation is simplified into the form of the equation (9). Then a second nonlinear feedback (equation 23) linearizes and decouples the input-output map. Therefore, the overall system is decoupled into two linear subsystems, where the distance control subsystem is of a second order, and the velocity control subsystem is of a first order. To stabilize the subsystems it becomes necessary to place the poles of the system, that is achieved with the external linear feedback loop.

4.3. Simulation results

In this section some computer simulation results regarding the control of the mobile robot dynamic model (presented in the previous section) are presented.

The kinematic parameters are similar to those of LABMATE platform: $a=2$, $b=0.75$, $d=0.3$, $r=0.15$, $m_c=30$, $m_w=1$, $I_c=15.625$, $I_w=0.005$ and $I_m=0.0025$. The initial velocity is 5 *m/s*, the initial reference point is $(x_c, y_c)=(32.0,\ 15.0)$, and $v_1^d=0$, $v_2^d=1.414$, $h_1(q)=\sqrt{(x_c-18.0)^2+(y_c-18.0)^2}-12.0$ and $h_2(v)=r/2(v_1+v_2)$.

The initial velocity is the condition that most affects the robot trajectory, therefore it is one of the most important parameters to consider in conjunction with initial heading angle (ϕ), as it can be seen in Figure 3(a) and (b).

Note that, depending on the gain values, obtained for the linear external loop (on the occasion of the pole placement) the mobile robot converges faster or more slowly to the desired path.

By observation of Figure 3(a), it can be concluded that for different initial heading angles (but with constant initial velocity of 5 *m/s*) the response is satisfactory. In the same

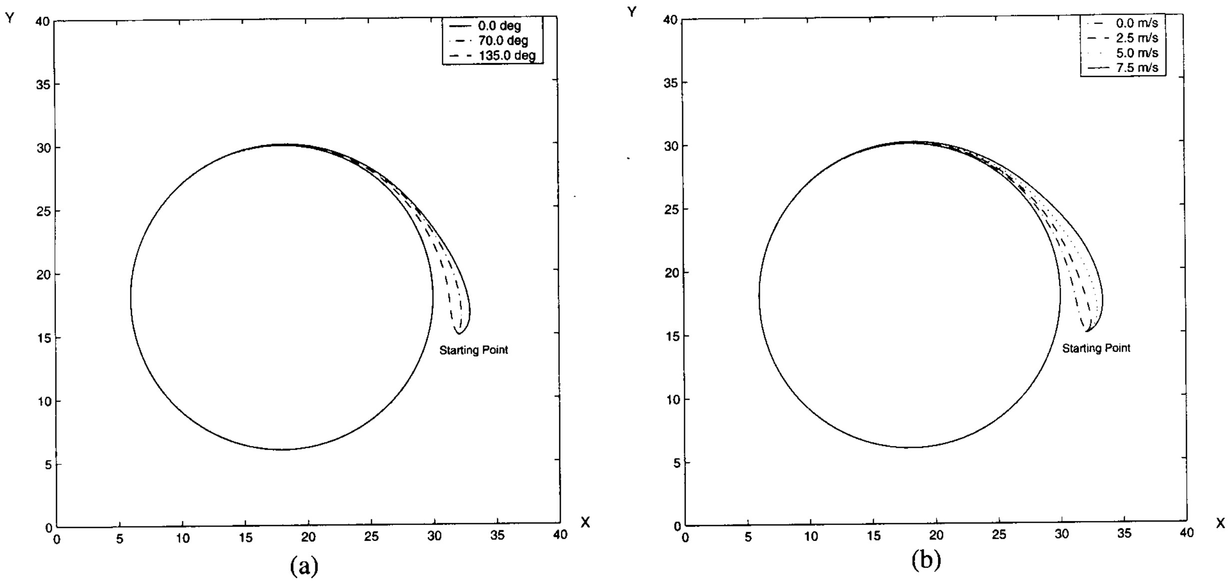

Fig. 3. Path following: (a) different initial heading angles for an initial forward velocity of 5 m/s; (b) different initial forward velocities for an initial heading angle of 0 degrees.

way for different initial velocities but with a constant heading angle (in this case 0 *degrees*), if the robot is far from the desired path, it is found (see Figure 3b)) that the system exhibits a better performance when the initial velocity is smaller.

5. CONCLUSION

In this paper we showed the application of the Lie algebra in the control of mobile robots for a system with holonomic and nonholonomic constraints. The application of Lie algebra was clear during the discussion of the input-output linearization and also of the zero-dynamics. However, to a real understanding of the use of this "tool" it became necessary to approach other issues such as the Lagrange dynamic equations, the nonlinear feedback, and the choice of the most adequate output variables for the type of control. Finally, after the implementation of the path following controller some results of computer simulation were presented to illustrate the performance of the algorithm, and to prove once again the contribution of Lie algebra to the design of appropriate controllers for mobile robots.

Acknowledgments

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APPENDIX A

Demonstration of equations (20 to 22):

Knowing that

$$z=T(x)=\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}=\begin{bmatrix} h(q) \\ L_f h(q) \\ \tilde{h}(q) \end{bmatrix}=\begin{bmatrix} h(q) \\ \Phi(q)v \\ \tilde{h}(q) \end{bmatrix}, \quad \dot{q}=S(q)v(t) \quad \text{and}$$

$J_h=\dfrac{\partial h}{\partial q}$ we calculate the derivatives of z as follows:

- $\dot{z}_1=\dfrac{\partial h}{\partial q}\,\dot{q}=J_h S(q)v$, giving $\dot{z}_1=\Phi(q)v=z_2$ knowing that $\Phi(q)=J_h(q)S(q)$.
- $\dot{z}_2=\dot{\Phi}(q)v+\Phi(q)\dot{v}$, and taking $\dot{v}=u$ in (9) results $\dot{z}_2=\dot{\Phi}(q)v+\Phi(q)u=\upsilon$.
- $\dot{z}_3=\dfrac{\partial \tilde{h}}{\partial q}\,\dot{q}\rightarrow \dot{z}_3=J_{\tilde{h}}S(q)v(t)$. To eliminate v in $\dot{z}_3$ equation and knowing that $z_2=\Phi(q)v \Rightarrow v=z_2\Phi^{-1}(q)$, and considering that $\dot{z}_3$ comes in terms of Jacobians, it is necessary to eliminate $\Phi(q)$ as follows:

$$\dot{z}_3=\frac{\partial \tilde{h}}{\partial q}\,\dot{q}\rightarrow \dot{z}_3=J_{\tilde{h}}S\Phi^{-1}(q)z_2, \text{ as } \Phi(q)=J_h S$$

$$\text{results } \dot{z}_3=J_{\tilde{h}}S(J_h S)^{-1}z_2.$$

Note that in this case z_3 and $\dot{z}_3$ represent the internal dynamic part of the system, that is the unobservable part.

APPENDIX B

Demonstration of calculation of s_3:

Knowing that the calculation of Lie brackets is attained through $s_3=[s_1, s_2]=\dfrac{\partial s_2}{\partial q}\,s_1-\dfrac{\partial s_1}{\partial q}\,s_2$, then:

$$\frac{\partial s_2}{\partial q}s_1=\begin{bmatrix}0&0&c(-b\sin\phi+d\cos\phi)&0&0\\0&0&c(b\cos\phi+d\sin\phi)&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}.$$

$$\begin{bmatrix}c(b\cos\phi-d\sin\phi)\\c(b\sin\phi+d\cos\phi)\\c\\1\\0\end{bmatrix}=\begin{bmatrix}c^2(-b\sin\phi+d\cos\phi)\\c^2(b\cos\phi+d\sin\phi)\\0\\0\\0\end{bmatrix}$$

$$\frac{\partial s_1}{\partial q}s_2=\begin{bmatrix}0&0&c(-b\sin\phi-d\cos\phi)&0&0\\0&0&c(b\cos\phi-d\sin\phi)&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}.$$

$$\begin{bmatrix}c(b\cos\phi+d\sin\phi)\\c(b\sin\phi-d\cos\phi)\\-c\\0\\1\end{bmatrix}=\begin{bmatrix}c^2(b\sin\phi+d\cos\phi)\\-c^2(b\cos\phi-d\sin\phi)\\0\\0\\0\end{bmatrix}$$

Attending to the fact that $s_3=\frac{\partial s_2}{\partial q}s_1-\frac{\partial s_1}{\partial q}s_2$ then

$$s_3=\begin{bmatrix}c^2(-b\sin\phi+d\cos\phi)-c^2(b\sin\phi+d\cos\phi)\\c^2(b\cos\phi+d\sin\phi)+c^2(b\cos\phi-d\sin\phi)\\0\\0\\0\end{bmatrix}=$$

$$\begin{bmatrix}-c^2(2b\sin\phi)\\c^2(2b\cos\phi)\\0\\0\\0\end{bmatrix}$$, and as $c=\frac{r}{2b}$ we obtain finally

$$s_3=\begin{bmatrix}-\frac{r}{2b}c(2b\sin\phi)\\\frac{r}{2b}c(2b\cos\phi)\\0\\0\\0\end{bmatrix}=\begin{bmatrix}-rc\sin\phi\\rc\cos\phi\\0\\0\\0\end{bmatrix}.$$

The demonstration for s_4, s_5, s_6, s_7 and s_8 are similar.

APPENDIX C

Definition *Stability: The equilibrium state* $\boldsymbol{x}=\mathbf{0}$ *of the dynamic system is said to be stable if, for any* $\boldsymbol{R}>\mathbf{0}$*, there exists* $\boldsymbol{r}>\mathbf{0}$*, such that if* $\|\boldsymbol{x}(\mathbf{0})\|<\boldsymbol{r}$*, then* $\|\boldsymbol{x}(t)\|<\boldsymbol{R}$ *for all* $t>0$*. Otherwise, the equilibrium point is unstable.*

The above definition of stability is also called Lagrange Stability or *Lyapunov Stability.*

Definition *Asymptotic Stability: An equilibrium point* $\mathbf{0}$ *is asymptotically stable if it is stable, and if in addition, there exist some* $\boldsymbol{r}>\mathbf{0}$ *such that* $\|\boldsymbol{x}(\mathbf{0})\|<\boldsymbol{r}$ *implies that* $\boldsymbol{x}(\boldsymbol{t})\rightarrow\mathbf{0}$ *as* $\boldsymbol{t}\rightarrow\infty$.

Asymptotic stability means that the equilibrium is stable, and that in addition, states started close to zero converge to zero as time goes to infinity. An equilibrium point which is Lagrange stable but not asymptotically stable is called *marginally stable*.

To a better understanding of the preceding definitions Figure 4 must be considered.

In Figure 4 **R** represents the radius of the larger circumference, **r** is the radius of the smallest circumference, $\mathbf{S_R}$ is the surface of the larger circle, $\mathbf{S_r}$ is the surface of the smallest circle and **0** is the center of the circumferences. By observation of the figure it can be concluded, regarding the systems stability, that:

- *Stable* – $x(0)$ must be always near **0** and its norm must be less than **r**;
- *Asymptotically stable* – Curve 1;
- *Marginally stable* – Curve 2;
- *Unstable* – Curve 3.

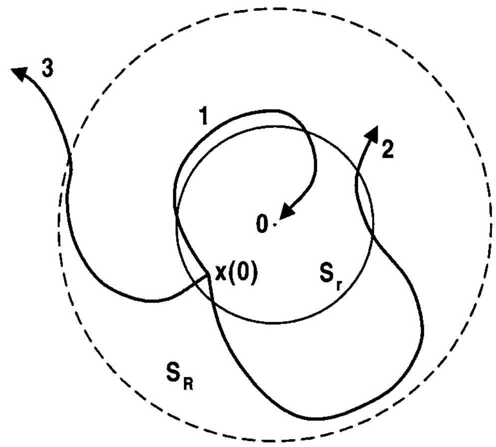

Fig. 4. Concepts of stability diagram.